# Mid-semestral Examination M. Math II Year (Differential Geometry II) 2014

Attempt all questions. Each question carries 5 + 5 (=10) marks. Books and notes maybe consulted Results proved in class, or propositions (with or without proof) from the class notes maybe used after quoting them. Results from exercises, however, must be proved in full if used. ).

1. Let $M$ be the subset of $\mathbb{R}^3$ defined by:
$$M = \{(x, y, z) : (x + y)^2 - z^2 = 3, \;\; x + 3y - 2z^7 = 0\}$$
   (i): Prove that $M$ is a smooth 1-dimensional submanifold of $\mathbb{R}^3$, and compute its tangent space at the point $a = (2, 0, 1)$.

   (ii): Let $f : M \to \mathbb{R}$ be the smooth map defined by $f(x, y, z) = x$. Is this map submersive at the point $a = (2, 0, 1)$?

2. Let $\pi : E \to M$ be a smooth vector bundle on a smooth manifold $M$, and let $s : M \to E$ be a smooth section of this bundle. Show that:

   (i): $s$ is an immersion.

   (ii): $s$ is a homeomorphism to its image, and hence an embedding.

3. View the torus $T^2$ as the surface obtained by revolving the circle of unit radius centred at $(2, 0)$ in the $xz$-plane, about the $z$-axis.

   (i): Compute the restiction of the 2-form $dx \wedge dz$ to $T^2$ in the coordinate chart defined by
$$\phi(s, t) = ((2 + \cos s) \cos t, (2 + \cos s) \sin t, \sin s)$$
   for $(s, t) \in (0, 2\pi) \times (0, 2\pi)$.

   (ii): Consider the regular 2-cube in $T^2$ defined by $\sigma(s, t) = \phi(2\pi s, 2\pi t)$ for $(s, t) \in [0, 1] \times [0, 1]$. Compute the integral:
$$\int_\sigma \omega$$
   where $\omega$ is the 2-form $(dx \wedge dz)_{|T^2}$.

4. Let $\phi_1 : (0, 2\pi) \to S^1 \setminus (0, 1)$ defined by $\phi_1(t) = (\cos t, \sin t)$ and $\phi_2 : (0, 2\pi) \to S^1 \setminus (-1, 0)$ defined by $\phi_2(s) = (-\cos s, -\sin s)$ be the usual two charts of a smooth atlas on $S^1$.

   (i): Construct a 1-form $\omega$ on $S^1$ whose pullbacks under these charts are: $\phi_1^*\omega = dt$ and $\phi_2^*\omega = ds$. (*Hint:* What is the coordinate change $\phi_1^{-1}\phi_2$ on $(0, \pi) \cup (\pi, 2\pi)$?)

   (ii): Prove that $\omega$ is not an exact form on $S^1$.